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NEW METHODS FOR SYNTHESIS OF
NONUNIFORMLY SPACED ANTENNA ARRAYS

TECHNICAL DOCUMENTARY REPORT NO. RADC-TDR-63-131

May 1963

Electronic Warfare Laboratory

Rome Air Development Center

Research and Technology Division

Air Force Systems Command

Griffiss Air Force Base, New York

Project No. 4540, Task No. 454001

FOREWORD

The research reported herein was performed in the Antenna
Laboratory, Department of Electrical Engineering, The Ohio State
University, under Contract No. AF30(602)-2711 between The Ohio
State University Research Foundation and Rome Air Development Center,
Air Force Systems Command, USAF, Griffiss AFB, N. Y. Mr. Karl Kirk
of Rome Air Development Center was the contract initiator.

This report (Contractor's No. 1423-1) was written by Dr. H. Unz
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ABSTRACT

Four new methods of synthesis of nonuniformly spaced antenna arrays are given:

    I. The mechanical quadratures method, developed recently by Bruce and Unz and independently by Lo.

    II. The eigenvalues method.

    III. The expansion method.

    IV. The orthogonalization method using the Schmidt's procedure.

An exponential-decay directive pattern is suggested in order to avoid numerical integrations. A summary of the research work in nonuniformly spaced antenna arrays is given.

PUBLICATION REVIEW

This report has been reviewed and is approved.

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TABLE OF CONTENTS

Page

v

NEW METHODS FOR SYNTHESIS OF
NONUNIFORMLY SPACED ANTENNA ARRAYS

1. INTRODUCTION

The importance of directive antennas was realized in the early days of radio communications. The principles of wave interference, on which systems of directive radio are based, has been known probably for several centuries. However, the first thorough treatment of this subject was conducted by Huygens and by Fresnel, who established the wave theory of light in the early part of the nineteenth century.

During the decade 1920-1930, a concentrated effort on directive properties of antenna arrays was started. During this period short-wave radio communications were becoming more popular, taking the place of long radio waves, and the use of antenna arrays of reasonable size became more feasible.

In 1937 Wolff[1] published his method of synthesizing any arbitrary far-zone circularly symmetric pattern with radiators uniformly distributed along an array axis. His theory was based upon comparison of the far-zone field of a pair of radiators to a term of the Fourier series expansion of the prescribed pattern.

During the second world war, the invention and the use of radar increased the interest in directive arrays. In 1943 Schelkunoff[2] utilized the correspondence between nulls of the pattern of a linear array having equidistant elements and the roots of a polynomial in the complex plane. Different types of pattern variations were derived by choosing different distributions of the zeros of the polynomial.

In 1946 Dolph[3] devised a method of synthesizing an optimum pattern of isotropic elements equally spaced in a uniform broadside linear array, in which the elements are fed in phase and are symmetrically arranged about the center of the array. The resultant current distribution across the array is based upon the properties of the Tchebyscheff polynomials and offers, from the design standpoint, much greater control of the pattern than previous theories.

Woodward[4] and Lawson[5] gave a method for calculating the field over a plane aperture to produce a given polar diagram and discussed the theoretical precision with which an arbitrary radiation pattern may be obtained from an array of finite size. This method is based on the idea of matching the radiation pattern of the aperture at a finite number of directions with the prescribed radiation pattern.

Van der Maas[6] gave the ideal Tchebyscheff space factor corresponding to a continuous source of finite length, in the following form:

$$(1a) \qquad F(\theta) = \cos\left(\pi \sqrt{u^2 - A^2}\right)$$

$$(1b) \qquad u = \frac{2a}{\lambda} \sin\theta$$

where $2a$ in the length of the line source, λ is the wavelength, and θ is the angle from the normal to the antenna axis. From the above it can be found that:

$$(1c) \qquad \mathrm{ch}(\pi A) = \text{side lobe ratio.}$$

By using Fourier integrals, Taylor[7] showed that the aperture current distribution required in order to produce the ideal radiation pattern in Eq. (1) is:

$$(2) \qquad g(p) = \frac{\pi A^2}{2} \cdot \frac{I_1\left(A\sqrt{\pi^2 - p^2}\right)}{A\sqrt{\pi^2 - p^2}}$$

$$+ \frac{1}{2}\,\delta(p - \pi) + \frac{1}{2}\,\delta(p + \pi), \quad p^2 \leq \pi^2$$

$$g(p) = 0, \qquad p^2 > \pi^2$$

where $p = \dfrac{\pi x}{a}$, δ is the Dirac delta function, and I_1 is the modified Bessel function. This pattern was considered later by Taylor[8] from the complex function theory point of view. Because of theoretical limitations, the above pattern cannot be obtained from a physically realizable antenna, but its ideal characteristics can be approached arbitrarily closely.

Cheng and Ma[8a] proposed a new approach to the uniformly spaced linear-array analysis. They considered the current distribution in the discrete elements of the linear array as the sampled data values of a continuous function, and used known relations in Z transforms developed for sampled-data systems in order to express the array polynomial in a closed form, where the array properties are easily found.

In all of the above cases, the spacings of the array elements are assumed to be uniform. In 1956 Unz[9,10,11] first introduced linear arrays with arbitrarily distributed elements and developed a theory for them. A relationship between the currents in the elements of the array, their distribution along the axis of the array, and the coefficients of the complex Fourier expansion of the radiation pattern were given in a matrix form which involved Bessel functions. It was pointed out that a nonuniformly spaced antenna array has more degrees of freedom than a similar uniform array with equally spaced elements, and as a result its performance should be better.

Further numerical work on the subject has been done by King, Packard and Thomas[12] , who showed that nonuniform antenna arrays with unequally spaced elements are also more broadband for different source frequencies. This is especially useful because of the earlier development of frequency-independent antenna elements by Rumsey, DuHamel, Isbell, and others[13]. Sandler[14] showed some equivalences between e' ally and unequally spaced arrays.

Andreasen[15] and several of his associates did extensive numerical work on digital and analog computers in order to find the general behavior of the radiation patterns of nonuniformly spaced arrays, in particular when the average spacing is larger than one wavelength. Swenson and Lo[16] considered the use of nonuniformly spaced arrays for large radio telescopes. Harrington[17] used perturbational procedures for reducing the side-lobe level of a nonuniformly spaced array with uniform excitation. Extensions of the previous theory to dipole elements

3

in the Fresnel zone[18], and to nonuniform spacings larger than one wavelength[19], were given by Unz. Bruce and Unz[20] gave a possible condition for broadbanding. Maffett[21] discussed array factors with a nonuniform spacing parameter by using well-known numerical integration techniques (trapezoidal, Simpson), and considered a statistical theory. Yen and Chow[21a] discussed the possibility of expressing the radiation pattern of large nonuniformly spaced arrays in closed form by using integration by stationary phase technique.

Pokrovskii[22] designed a 4- and 6- element array having nonuniform spacing and uniform excitation of the elements, with improved patterns over the Dolph-Tchebyscheff array for the same length and the same number of elements. While his method of solution is general, the transcendental equations become more and more involved for larger arrays. Additional discussion has been given by Brown[34].

Nonuniformly spaced arrays have several advantages over uniformly spaced arrays; namely, their performance can be better, fewer elements can be used, and they are more broadband. However, nonuniformly spaced arrays are seldom used in practice at present. Besides being latecomers, the great difficulty in synthesizing such arrays seem to be the main reason for the hesitancy to use them.

Recently a new method of synthesizing radiation patterns using nonuniformly spaced arrays has been developed by Bruce and Unz[23,24]. They applied a mechanical quadratures formulation in order to transform a continuous aperture distribution to a nonuniformly spaced array. Because of its importance, a summary of this work will be given in Section 3 of this paper. Some work along similar lines has been done independently by Lo[25].

The aim of this paper is to give new methods for synthesis of nonuniformly spaced antenna arrays. It is hoped that these new methods will be found useful in the design of nonuniformly spaced arrays and helpful in achieving the general use of their full potentialities.

2. THE EXPONENTIAL PATTERN

One of the possible directive radiation patterns with exponential decay, which will be used later on, can be written in the form:

$$(3) \qquad F(\theta) = e^{-a^2 \sin^2 \theta} \cos (2b \sin \theta)$$

where θ is the angle of the radius vector from the normal to the array axis, and "a", "b" are arbitrary constants. The maximum value of the pattern in the direction normal to the array axis is normalized, $F(0) = 1$.

The first null of the radiation pattern will be at $\theta = \theta_o$ such that

$$(4a) \qquad 2b \sin \theta_o = \frac{\pi}{2}$$

and the beamwidth w_o between the first nulls will be $w_o = 2\theta_o$; one obtains, therefore,

$$(4b) \qquad b = \frac{\pi}{4 \sin \frac{w_o}{2}}$$

Using Eq. (4b), the constant "b" can be determined if the beamwidth between the nulls w_o is specified.

Assuming the first side-lobe maximum of the radiation pattern to be at $\theta = \theta_1$, then the side-lobe level M of the radiation pattern will be defined by:

$$(5a) \qquad M = \frac{F(0)}{|F(\theta_1)|} = \frac{1}{|F(\theta_1)|} = \frac{e^{a^2 \sin^2 \theta_1}}{|\cos (2b \sin \theta_1)|}$$

The side-lobe level in decibels M_{db} will be then:

$$(5b) \qquad M_{db} = 20 \log M =$$

$$= 20 \left[a^2 \sin^2 \theta_1 \log e - \log |\cos (2b \sin \theta_1)| \right].$$

5

In order to find the position of the first maximum, one should solve $F'(\theta_1) = 0$. Using Eq. (3) one obtains:

$$b \sin(2b \sin \theta_1) + a^2 \sin \theta_1 \cos(2b \sin \theta_1) = 0.$$

Using the notation $\alpha = 2b \sin \theta_1$ one gets:

$$(6) \qquad \frac{tg \, \alpha}{\alpha} = - \frac{a^2}{2b^2} .$$

Using Eq. (6) and the same notation, Eq. (5b) becomes:

$$(7) \qquad \frac{M_{db}}{20} = - 0.217 \, \alpha \, tg \, \alpha - \log \left| \cos \alpha \right|$$

Equations (6) and (7) may be used in order to find the relationship between the constants "a", "b", and the side-lobe level M_{db}, and the results are shown graphically in Figures 1, 2, 3.

The required radiation pattern is given usually in terms of its beamwidth w_o and the side-lobe level M_{db}. From Eq. (4b) the constant "b" of the radiation pattern can be found from the beamwidth w_o. For a given side-lobe level M_{db}, the transcendental equation (7) can be solved for α. This could be accomplished simply by using Figure 2. Then Eq. (6) and the value of the constant "b" should be used in order to find the constant "a". Figure 1 could be helpful in this case. Figure 3, which is a combination of the previous figures, gives a direct relationship, in a graphical form, between the side-lobe level M_{db} and the constant $\dfrac{a^2}{2b^2}$.

We have seen that if the beamwidth w_o and the side-lobe level M_{db} are given, the constants "a", "b" may be found and the directive, exponential-decay radiation pattern in Eq. (3) will be completely specified.

3. MECHANICAL QUADRATURES METHOD

Recently a new method of synthesis of nonuniformly spaced linear arrays has been developed by Bruce and Unz[23,24] and independently by Lo[25]. They applied mechanical quadratures[26] in order to transform

$$-\frac{tg\,\alpha}{\alpha} = \frac{a}{2b^2}$$

Fig. 1.

7

Fig. 2.

8
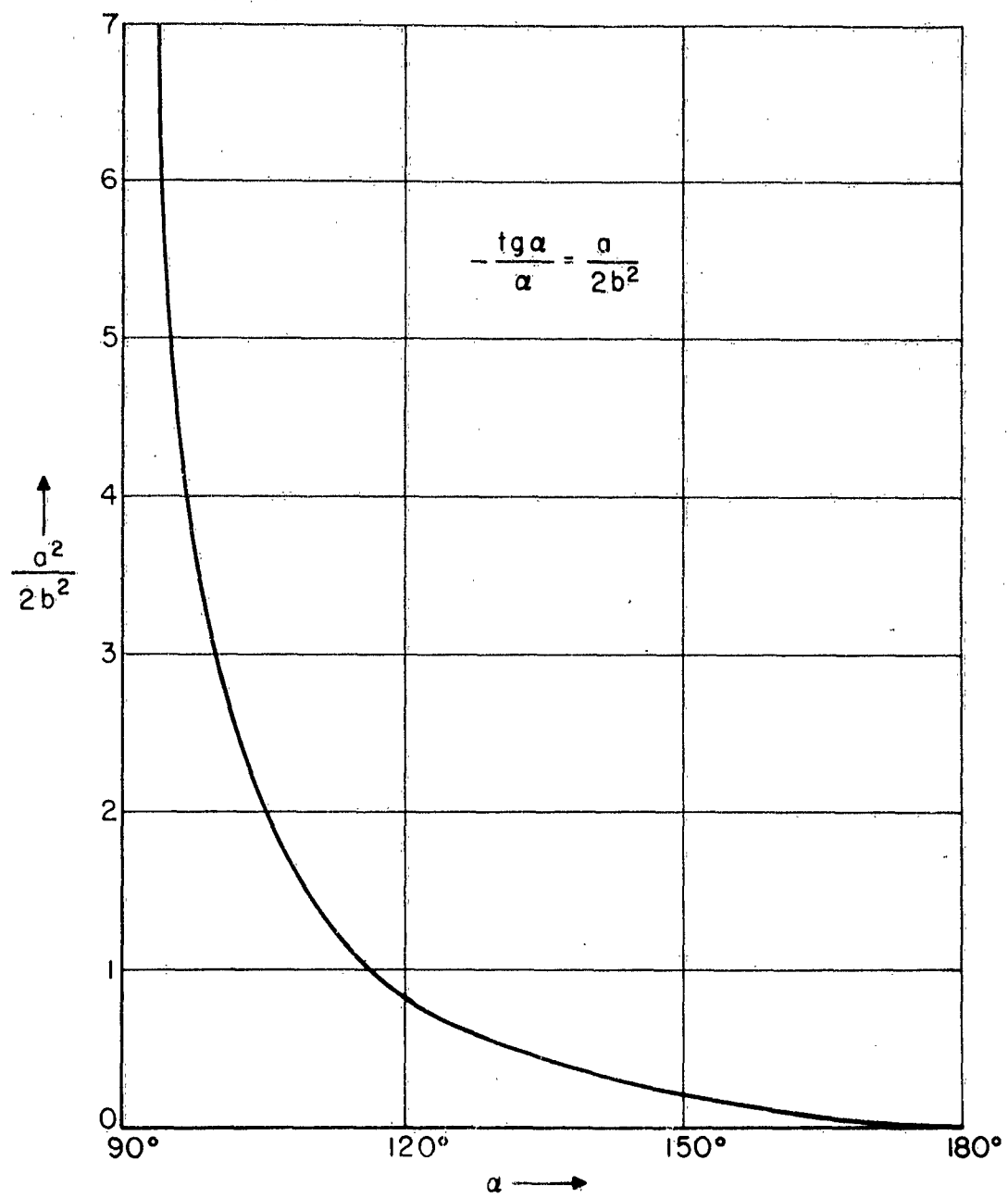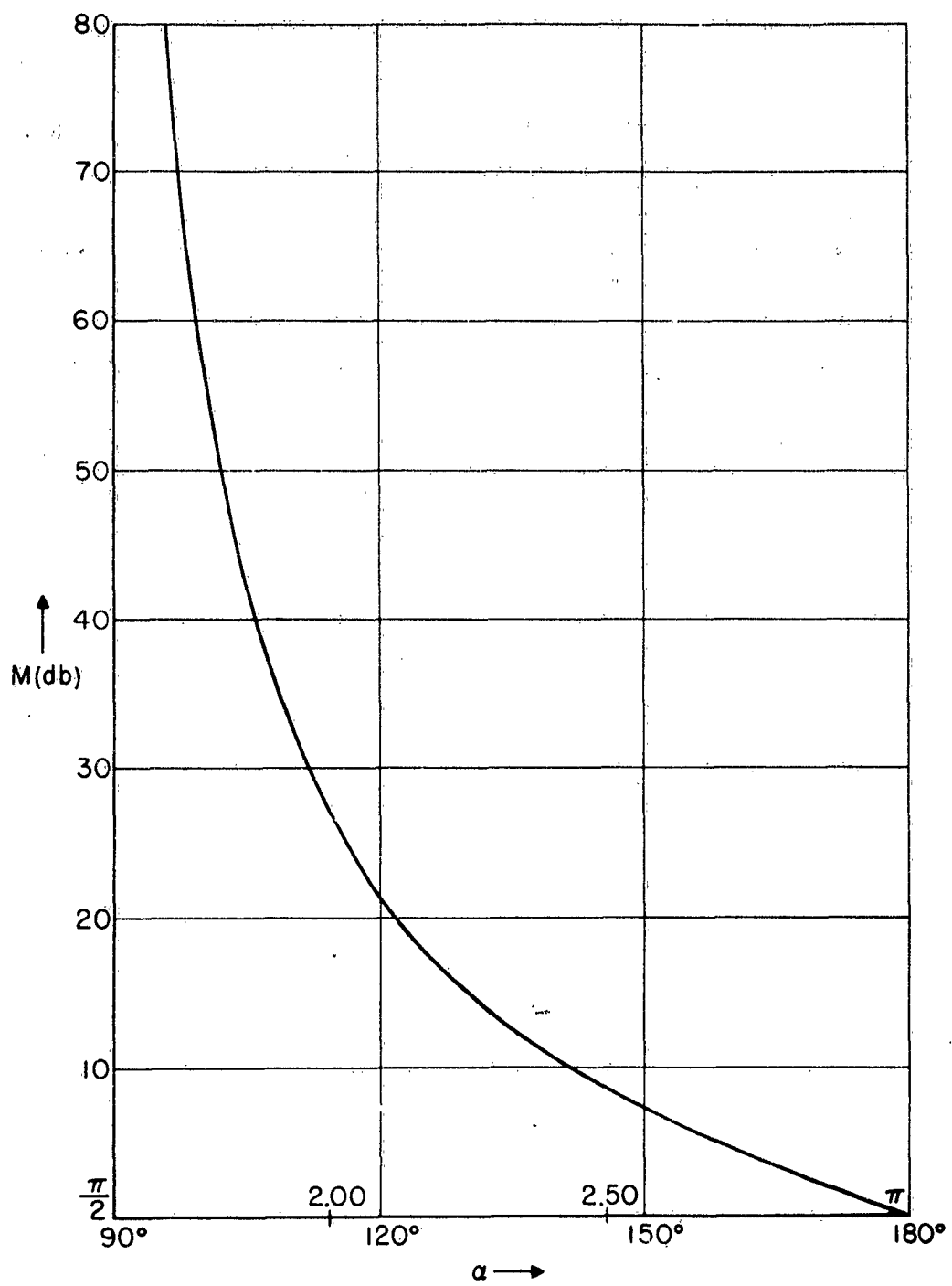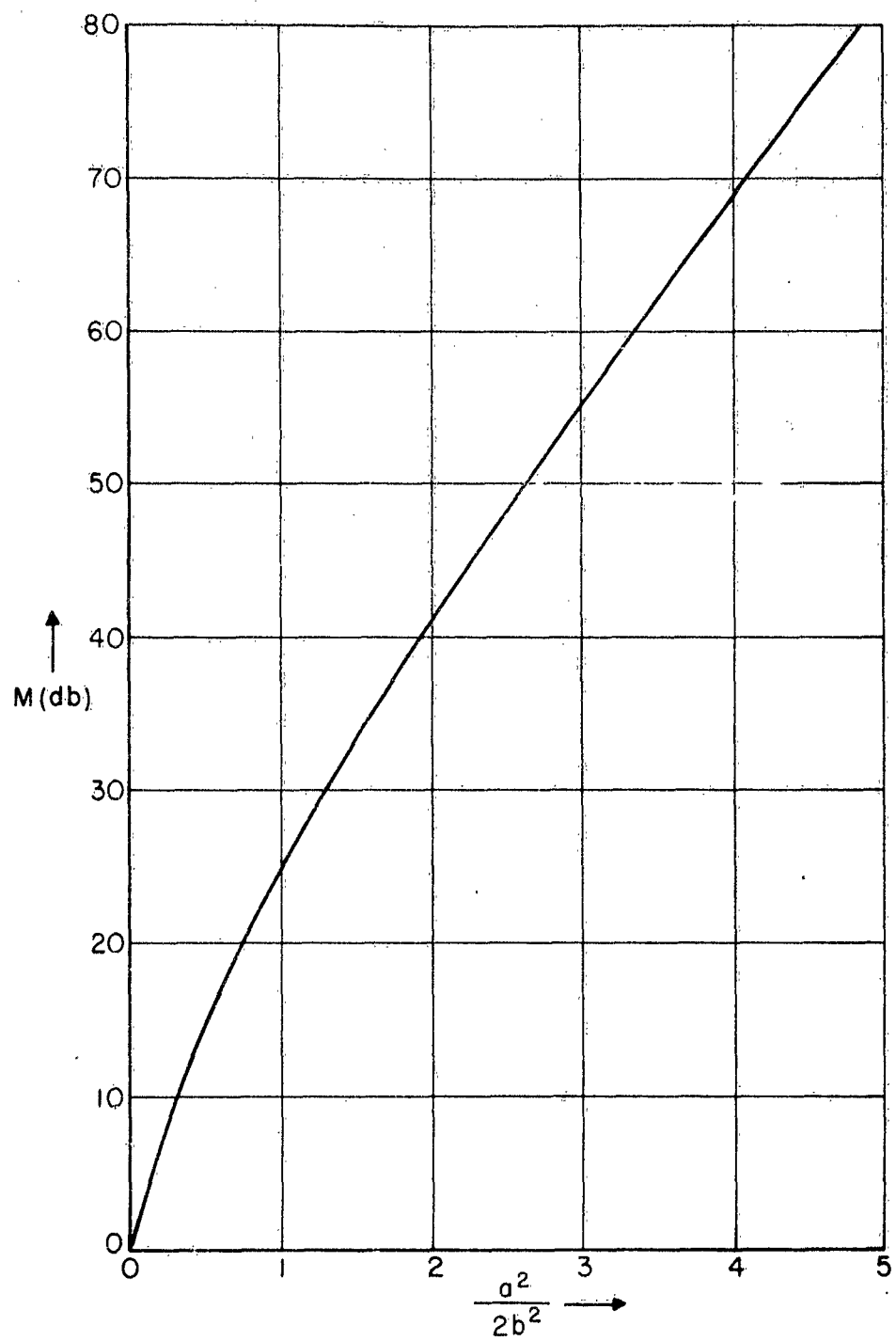
Fig. 3.

a continuous aperture distribution to a nonuniformly spaced array. Because of its importance, a summary of this new method [23],[24] is presented here.

Using weight functions, the general form of mechanical quadratures[26] is given by:

$$(8) \qquad \int_{-1}^{+1} w(x) f(x) \, dx = \sum_{j=1}^{n} H_j f(a_j) \, .$$

The n values of a_j are real, distinct, within the interval $(-1, +1)$, and are the roots of a polynomial orthogonal with respect to the weight function $w(x)$ on the interval $(-1, +1)$. The n values of H_j are real, positive, and can be determined by solving the set of linear equations generated by letting $f(x) = x^k$ for $k = 0, 1, 2 \cdots 2n - 1$. The only requirements on the weight function are that it does not vanish within the interval of integration and that it is integrable.

For a finite aperture with symmetric excitation, the radiation pattern can be written as:

$$(9) \qquad F(u) = \int_{-1}^{+1} g(x) \cos u \, x \, dx$$

where the aperture length has been normalized, $u = \pi \sin\theta$, and x is the distance from the origin measured in half wavelengths.

Comparing Eqs. (8) and (9), and taking the weight function to be equal to the aperture distribution $w(x) \equiv g(x)$, one obtains:

$$(10) \qquad F(u) = \sum_{j=1}^{n} H_j \cos (u \, a_j) \, .$$

Equation (10) is identified as the nonuniform symmetric array radiation pattern, where the element positions are given by the n values of a_j and the current excitation coefficients are given by n values of H_j.

10

In case of no weight function, $w(x) = 1$, Eq. (8) becomes:

$$(11) \qquad \int_{-1}^{+1} f(x)\, dx = \sum_{j=1}^{n} H_j f(a_j)$$

and as a result, Eq. (10) becomes:

$$(12) \qquad F(u) = \sum_{j=1}^{n} H_j g(a_j) \cos(u a_j) \ .$$

Bruce and Unz[23,24] applied this method to the exponential directive radiation pattern described in Section 2, as well as to the Dolph-Tchebyscheff pattern, with very good results. In the case of the exponential pattern, by taking the weight function to be $w(x) = e^{-x^2}$ one finds the values of a_j and H_j to be tabulated[26].

The basic requirement for the mechanical quadratures method is the knowledge of the aperture distribution $g(x)$ in Eq. (9). If the radiation pattern $F(u)$ is prescribed, $g(x)$ could be found sometimes by using the inverse Fourier transform, as for the case of the exponential pattern[23,24] in Section 2. Sometimes the aperture distribution $g(x)$ may be found by taking the envelope[27] of an already designed uniformly-spaced array, as for the case of the Dolph-Tchebyscheff pattern[23,24]. In both cases the current at the aperture edge will give difficulty, as can be seen in Eq. (2), for example, and should be discussed separately[8,28].

For the design of a general prescribed radiation pattern, the difficulties of this method are twofold: (a) To find the aperture excitation from the prescribed radiation pattern by a Fourier integral; (b) Approximating the aperture excitation by a polynomial, a process which will involve the inversion of a matrix. However, for directive patterns certain short cuts in the process have been shown above.

4. EIGENVALUES METHOD

A symmetric, nonuniformly spaced antenna array will give the radiation pattern:

$$(13) \qquad F(u) = \sum_{\ell=0}^{L} A_\ell \cos(u x_\ell)$$

where $u = \pi \sin\theta$, θ being the angle with the normal to the array axis, and x_ℓ is the distance from the center, measured in half wavelengths. In Eq. (13) the radiation pattern $F(u)$ is given and the array element distribution x_ℓ and the element amplitudes A_ℓ have to be determined.

Let us discuss the integral:

$$(14a) \qquad I(x_\ell; x_m) = \int_{-\pi}^{+\pi} \cos(u x_\ell) \cos(u x_m) \, du.$$

When $x_\ell = x_m$ one obtains:

$$(14b) \qquad I_\ell = I(x_\ell; x_\ell) = \pi\left[1 + \frac{\sin 2 x_\ell \pi}{2 x_\ell \pi} \right].$$

By using trigonometric identities Eq. (14a) becomes:

$$I(x_\ell; x_m) = \frac{\sin(x_\ell + x_m)\pi}{x_\ell + x_m} + \frac{\sin(x_\ell - x_m)\pi}{x_\ell - x_m}.$$

Equating the last expression to zero leads to:

$$(x_\ell - x_m) \sin(x_\ell + x_m)\pi + (x_\ell + x_m) \sin(x_\ell - x_m)\pi = 0.$$

Rearranging,

$$x_\ell [\sin(x_\ell + x_m)\pi + \sin(x_\ell - x_m)\pi] =$$

$$= x_m [\sin(x_\ell + x_m)\pi - \sin(x_\ell - x_m)\pi] \ .$$

Using trigonometric identities, one obtains finally:

$$(14c) \qquad (x_\ell \pi)\, \mathrm{tg}\,(x_\ell \pi) = (x_m \pi)\, \mathrm{tg}\,(x_m \pi) \ .$$

It is found that for the eigenvalues of Eq. (14c), the integral $I(x_\ell; x_m)$ in Eq. (14a) will have orthogonality properties such that:

$$(14d) \qquad I(x_\ell; x_m) = \pi \left[1 + \frac{\sin 2 x_\ell \pi}{2 x_\ell \pi} \right] \delta_{\ell, m}$$

where

$$\delta_{\ell, m} = \begin{cases} 1 & \ell = m \\ 0 & \ell \neq m \end{cases}$$

is the Kronecker delta.

From Eq. (14), for the eigenvalues one can see that all the positions of the radiating elements are determined by the position of the first element with respect to the middle of the array. In order to include the first eigenfunction in the complete set, the first element position should be $0 \leq x_0 < 1$; in other words, the distance between the two innermost elements should be smaller than one wavelength. The eigenvalues in general are unequally spaced for small values and are almost equally spaced, a half wavelength apart, for large values.

Using the orthogonality properties in Eqs. (14) one may obtain from Eq. (13):

$$(15) \qquad A_\ell = \frac{1}{\pi} \left[1 + \frac{\sin 2 x_\ell \pi}{2 x_\ell \pi} \right]^{-1} \int_{-\pi}^{+\pi} F(u) \cos(u x_\ell)\, du \ .$$

13

Thus, one could get a nonuniformly spaced array which will produce any required radiation pattern. Its element distribution will be determined by Eq. (14c) and the currents will be determined by Eq. (15).

For the particular case of the exponential pattern, Eq. (3) may be rewritten:

$$(16) \quad F(u) = e^{-\frac{a^2}{\pi^2} u^2} \cos\left(\frac{2b}{\pi} u\right)$$

where $u = \pi \sin\theta$. Substituting Eq. (16) into Eq. (15) and using trigonometric identities, one may obtain definite integrals, which may be evaluated approximately[29] for $a \geq 2$:

$$(17) \quad \int_0^\pi e^{-\frac{a^2}{\pi^2} u^2} \cos(2pu)$$

$$\approx \int_0^\infty e^{-\frac{a^2}{\pi^2} u^2} \cos(2pu)\, du = \frac{\pi\sqrt{\pi}}{2a} e^{-\frac{\pi^2 p^2}{a^2}}$$

Using Eq. (17) one may obtain for the exponential pattern:

$$(18a) \quad A_\ell = \frac{\sqrt{\pi}}{2a} \left[1 + \frac{\sin 2x_\ell \pi}{2x_\ell \pi}\right]^{-1} \left[e^{-\left(\frac{b+\frac{\pi}{2}x_\ell}{a}\right)^2} + e^{-\left(\frac{b-\frac{\pi}{2}x_\ell}{a}\right)^2} \right],$$

or in alternative form:

$$(18b) \quad A_\ell = \frac{\sqrt{\pi}}{2a}\left[1 + \frac{\sin 2x_\ell \pi}{2x_\ell \pi}\right]^{-1} e^{-\frac{b^2}{a^2}} e^{-\left(\frac{\pi x_\ell}{2a}\right)^2} \text{ch}\left(\frac{b\pi x_\ell}{a^2}\right).$$

Equations (18) give the current distribution of an antenna array which produces the exponential radiation pattern in Eq. (3) or (16), and the elements of the array are distributed according to the eigenvalues of Eq. (14c).

From Eq. (18a) one can see that for

(19) $\qquad \frac{\pi}{2} x_\ell > a + b$

the value of A_ℓ quickly becomes very small, and the contributions of these terms to the radiation pattern are so small that they could be neglected. Equation (19) gives an indication of the number of elements required to produce the exponential pattern within a certain approximation.

For the particular condition of uniformly spaced arrays, one could distinguish between two cases:

A. Odd number of elements:

In this case $x_0 = 0$ and from Eq. (14c) one obtains $x_\ell = \ell$. Equation (15) becomes then:

(20a) $\qquad A_\ell = \frac{1}{\pi} \int_{-\pi}^{+\pi} F(u) \cos(\ell u) \, du \qquad\qquad \ell > 0$

(20b) $\qquad A_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} F(u) \, du$

B. Even number of elements:

In this case $x_0 = \frac{1}{2}$ and from Eq. (14c) one obtains $x_\ell = \ell + \frac{1}{2}$. Equation (15) becomes then:

(21) $\qquad A_\ell = \frac{1}{\pi} \int_{-\pi}^{+\pi} F(u) \cos\left(\ell + \frac{1}{2}\right) u \, du$

15

and the relationships for the exponential pattern should be changed accordingly in Eqs. (18).

In the last two methods of Sections 3 and 4 the positions of the array elements have been determined during the synthesis procedures. These methods will fail in general when the element positions are pre-assigned. In the next two sections we will discuss cases of this kind.

5. EXPANSION METHOD

When the radiation pattern $F(u)$ is given and the nonuniformly distributed antenna array element positions are preassigned, the determination of the currents A_ℓ in Eq. (13) is relatively difficult. The reason is that one is required to expand a given function in terms of a non-orthogonal set of functions. This problem has been discussed by Kantorovich and Krylov[30].

One possible method of evaluating the coefficients A_ℓ in Eq. (13) will be to take any complete set of functions $\{\psi_m(u)\}$, multiply both sides of Eq. (13) by $\psi_m(u)$ and integrate. One will obtain then a system of equations:

$$(22a) \qquad \sum_{\ell=0}^{L} c_{\ell,m} A_\ell = b_m \qquad\qquad m = 0,\ 1,\ 2 \cdots L$$

where:

$$(22b) \qquad c_{\ell,m} = \int_{-\pi}^{+\pi} \cos(u x_\ell)\, \psi_m(u)\, du$$

$$(22c) \qquad b_m = \int_{-\pi}^{+\pi} F(u)\, \psi_m(u)\, du$$

where $c_{\ell,m}$ and b_m are known and A_ℓ is to be evaluated by solving $(L+1)$ linear equations with $L+1$ unknowns.

16

It will be useful to indicate also another method of obtaining system (22a), which is doubtless even more convenient. Let us expand Eq. (13) in terms of any orthonormal system $\psi_m(u)$. Let:

$$(23a) \qquad F(u) = \sum_{m=0}^{\infty} b_m \psi_m(u)$$

$$(23b) \qquad \cos(u x_\ell) = \sum_{m=0}^{\infty} c_{\ell,m} \psi_m(u)$$

where b_m is given by Eq. (22c) and $c_{\ell,m}$ is given by Eq. (22b). Substituting Eqs. (23) into Eq. (13) and equating coefficients of $\psi_m(u)$ in both sides, one obtains Eq. (22a). We note that the second method of obtaining system (22a) presupposes, as distinct from the first, a system of functions $\psi_m(u)$ that is orthogonal and normalized.

One possible such set is $\psi_m(u) = \cos mu$. This set will have a definite advantage, as Eq. (22a) will be simplified considerably if some of the elements of the array are uniformly distributed at multiples of a half wavelength apart. Also, the integrals (22b) are readily available.

The above method could be extended to the case of asymmetric nonuniformly spaced arrays:

$$(24) \qquad F(u) = \sum_{\ell=0}^{L} A_\ell \, e^{i u x_\ell} .$$

One could take then $\psi_m(u) = e^{i m u}$. Repeating the above process, one will obtain:

$$(25a) \qquad \sum_{\ell=0}^{L} c_{\ell,m} A_\ell = b_m \qquad\qquad m = 0, 1, 2 \cdots L$$

$$(25b) \qquad c_{\ell,m} = \int_{-\pi}^{+\pi} e^{i u x_\ell} e^{i m u} \, du = 2\pi \frac{\sin \pi(x_\ell + m)}{\pi(x_\ell + m)}$$

17

$$(25c) \quad b_m = \int_{-\pi}^{+\pi} F(u) \, e^{i m u} \, du \ .$$

In this case, in general, b_m and A_ℓ will be complex and one should take both $+(m)$ and $-(m)$. The theory developed by Unz[9,10,11] originally is a particular case of the method described in this section.

6. ORTHOGONALIZATION METHOD

In order to expand an arbitrary function in terms of a nonorthogonal set of functions, the orthogonalization procedure by Schmidt may be used. This procedure has been discussed in detail by Kantorovich and Krylov[30], and its details for complex functions may be found in Appendix A of the present paper.

It may be shown[30] that from an infinite set of nonorthogonal complex functions:

$$\phi_1(u), \quad \phi_2(u) \cdots$$

an orthonormal set of functions

$$\psi_1(u), \quad \psi_2(u) \cdots$$

may be derived by using the relationships:

$$(26a) \quad \psi_1(u) = \frac{\phi_1(u)}{\left[\int_a^b \phi_1(u) \, \phi_1^*(u) \, du \right]^{\frac{1}{2}}}$$

$$(26b) \quad \psi_{n+1}(u) = \frac{\phi_{n+1}(u) - \sum_{j=1}^{n} \left(\int_a^b \phi_{n+1} \psi_j^* \, du \right) \psi_j(u)}{\left\{ \int_a^b \left| \phi_{n+1}(u) - \sum_{j=1}^{n} \left(\int_a^b \phi_{n+1} \psi_j^* \, du \right) \psi_j(u) \right|^2 \, du \right\}^{\frac{1}{2}}}$$

where $AA^* = |A|^2$. The orthonormal functions $\psi_2(u)$, $\psi_3(u)$, $\cdots \psi_{n+1}(u)$ may be found in succession.

The orthonormal set $\psi_n(u)$ obeys:

$$(27) \qquad \int_a^b \psi_\ell(u)\, \psi_m^*(u)\, du = \delta_{\ell, m} = \begin{cases} 1 & \ell = m \\ 0 & \ell \neq m \end{cases}$$

Let us assume that a set of nonorthogonal functions is given by:

$$(28) \qquad \phi_n(u) \quad e^{i x_n u} \qquad\qquad n = 1, 2, \cdots L$$

where $x_1 < x_2 < x_3 < \cdots < x_L$. We would like to orthogonalize this set of functions, in the region $(-\pi, +\pi)$, where the orthonormal set will be denoted by $\psi_n(u)$. From (28) and (26a) one obtains:

$$(29a) \qquad \psi_1(u) = \frac{1}{\sqrt{2\pi}}\, e^{i x_1 u}.$$

Let us define the following function:

$$(29b) \qquad \int_{-\pi}^{+\pi} e^{i(x_\ell - x_m)u}\, du = \left. \frac{e^{i(x_\ell - x_m)u}}{i(x_\ell - x_m)} \right|_{-\pi}^{+\pi}$$

$$= 2\pi\, \frac{\sin(x_\ell - x_m)\pi}{(x_\ell - x_m)\pi} = 2\pi S_{\ell m} = 2\pi S_{m\ell}$$

where $S_{\ell\ell} = 1$.

Using Eqs. (28), (29), and Eq. (26b), for $n = 1$, (one) obtains:

$$(29c) \qquad \psi_2(u) = \frac{1}{\sqrt{2\pi}}\, \frac{e^{i x_2 u} - S_{12}\, e^{i x_1 u}}{[1 - S_{12}^2]^{\frac{1}{2}}}$$

19

Similarly, one may find in Eq. (26b) after rearranging:

(29d) $\quad \psi_3(u) =$

$$\frac{1}{\sqrt{2\pi}} \frac{(1 - S_{12}^2)e^{i x_3 u} - (S_{32} - S_{12}S_{13})e^{i x_2 u} - (S_{13} - S_{12}S_{23})e^{i x_1 u}}{[1 - S_{12}^2]^{\frac{1}{2}} [1 - S_{12}^2 - S_{13}^2 - S_{23}^2 + 2S_{12}S_{13}S_{23}]^{\frac{1}{2}}}$$

By continuing the above process in Eq. (26b) one may find formally:

(29e) $\quad \psi_4(u) = c_1^{(4)} e^{i x_1 u} + c_2^{(4)} e^{i x_2 u} + c_3^{(4)} e^{i x_3 u} + c_4^{(4)} e^{i x_4 u}$

or in general:

(29f) $\quad \psi_n(u) = \displaystyle\sum_{\ell=1}^{n} c_\ell^{(n)} e^{i x_\ell u}$

where the constants $c_\ell^{(n)}$ may be found by the Schmidt orthogonalization procedure described above.

In the case of a general, asymmetric nonuniformly spaced array, one has:

(30) $\quad F(u) = \displaystyle\sum_{\ell=1}^{L} A_\ell e^{i x_\ell u}$

Taking the orthogonal set of functions $\psi_n(u)$ described in Eqs. (29) one could expand:

(31a) $\quad F(u) = \displaystyle\sum_{n=1}^{N} B_n \psi_n(u)$

20

where, using the orthogonality relationship in Eq. (27), one can find:

$$(31b) \qquad B_n = \int_{-\pi}^{+\pi} F(u)\, \psi_n^*(u)\, du \; .$$

Since $F(u)$ is prescribed, B_n may be evaluated from Eq. (31b) using Eqs. (29).

Substituting Eq. (29f) into Eq. (31a) one obtains, after rearranging:

$$(32) \qquad F(u) = \sum_{n=1}^{N} \sum_{\ell=1}^{n} B_n c_\ell^{(n)} e^{i x_\ell u} \; .$$

Comparing Eqs. (32) and (30) one obtains:

$$(33a) \qquad A_1 = B_1 c_1^{(1)} + B_2 c_1^{(2)} + B_3 c_1^{(3)} + \cdots B_N c_1^{(N)}$$

$$(33b) \qquad A_2 = \qquad\qquad B_2 c_2^{(2)} + B_3 c_2^{(3)} + \cdots B_N c_2^{(N)}$$

$$(33c) \qquad A_3 = \qquad\qquad\qquad\qquad B_3 c_3^{(3)} + \cdots B_N c_3^{(N)}$$

$$\cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot$$

$$(33d) \qquad A_N = \qquad\qquad\qquad\qquad\qquad\qquad B_N c_N^{(N)}$$

By using the above procedure one can see that in principle the non-uniformly spaced array can be designed as well as uniformly spaced arrays without using the process of inversion of large matrices. The process of finding the orthogonal set of functions in general is rather involved algebraically, but they could be found with the help of a digital computer using the successive process described by Eqs. (26). In general the result will correspond to the radiation pattern within a certain approximation, rather than be equal to it.

The great importance of this process of solution of nonuniformly spaced arrays is that it is especially designated for asymmetric non-uniformly spaced arrays, which have more degrees of freedom than symmetric nonuniformly spaced arrays. Of course, symmetric arrays also can be designed by using this method.

21

For the case of an exponential pattern described in Section 2, one is able to use Eqs. (16) and (17) in order to integrate in Eq. (31b) and find the corresponding coefficients B_n.

For the particular case of uniformly spaced arrays with $\frac{\lambda}{2}$ spacings, one will have in Eq. (29b):

(34a) $\quad x_\ell - x_m = \ell - m \; ; \qquad S_{\ell m} = \delta_{\ell, m}$

and Eqs. (29) become:

(34b) $\quad \psi_1(u) = \dfrac{1}{\sqrt{2\pi}} \; e^{i u}$

(34c) $\quad \psi_2(u) = \dfrac{1}{\sqrt{2\pi}} \; e^{i 2 u}$

(34d) $\quad \psi_3(u) = \dfrac{1}{\sqrt{2\pi}} \; e^{i 3 u}$

(34e) $\quad \psi_n(u) = \dfrac{1}{\sqrt{2\pi}} \; e^{i n u}$

and Eqs. (31), (32) and (33) will reduce to the standard Fourier series analysis of uniformly spaced arrays.

7. SUMMARY

In this paper the published research work in nonuniformly spaced antenna arrays has been summarized to date. Four new methods for

22

synthesis of nonuniformly spaced antenna arrays for a given radiation pattern have been suggested:

I. The mechanical quadratures method, developed recently by Bruce and Unz[23,24] and independently by Lo[25] is summarized.

II. The eigenvalues method, where the positions of the elements are given as eigenvalues of a transcendental equation.

III. The expansion method, which involves the inversion of matrices in order to find the currents of the elements.

IV. The orthogonalization method by Schmidt's procedure.

In methods I and II the positions of the elements are predetermined by the weight function involved (method I), or by the position of the innermost element (method II). In methods III and IV the array elements are first arbitrarily distributed (in accordance with the prerequisites, e.g., broadbanding) and then the current distributions in the elements are determined. Methods I and III will involve in general the inversion of a matrix, while in methods II and IV this is avoided.

An exponential-decay radiation pattern is suggested and used as an example in the above methods. Its main advantage in the present synthesis is that the definite integrals involved may be approximately evaluated explicitly, and thus simplify the numerical work. This directive exponential pattern could also be used for electronic scanning[31] problems in the form:

$$(35) \quad F(\theta - \gamma) = e^{-a^2 \sin^2 (\theta - \gamma)} \cos [2 b \sin (\theta - \gamma)]$$

where $\gamma = \gamma(t)$ is the angle direction in which the radiation pattern will have a maximum and $\gamma(t)$ is a function of time. The above methods could be modified slightly in order to calculate the required current distribution by numerical integration. The linear phase delay case for scanning may be found as a particular case under certain approximations.

Some numerical work using the above methods has been done, and it will be published in the future together with additional numerical work and comparisons between the four methods of synthesis, with regard to the best approximations.

Sharp[32] and Willey[33] suggested methods of design of linear and planar arrays in order to reduce the number of the elements required for a specific radiation pattern. Ishimaru[35] has recently suggested the use of the Poisson's sum formula for the design of nonuniformly spaced arrays. His method is useful in treating nonuniform arrays with large number of elements and unequally spaced arrays on curved surfaces.

It is hoped that by using the synthesis methods suggested by the author in the present paper and others, the design of the nonuniformly spaced arrays will become simpler and their use more generally accepted.

APPENDIX

In the following we will extend the orthogonalization process of Schmidt, as given by Kantorovich and Krylov,[30] to a set of complex functions.

Let there be given an infinite system of complex functions,

(A-1) $\phi_1(u)$, $\phi_2(u)$, $\cdots$

defined and continuous in the interval (a, b). We can exclude from the given system of functions those that represent linear combinations of the preceding ones, since by their nature they do not extend the system. Let us now carry through the orthogonalization of the system. (A-1), i.e., let us now construct the orthonormal system of functions

(A-2) $\psi_1(u)$, $\psi_2(u)$, $\cdots$

such that

(A-3) $$\int_a^b \psi_\ell(u)\, \psi_m^*(u)\, du = \delta_{\ell,m}$$

where $\delta_{\ell,m}$ is the Kronecker delta, and $\psi_m^*(u)$ is the complex conjugate function. Each function of Eq. (A-2) is to represent some linear combination of the functions of system (A-1), i.e., $\psi_n(u)$ will have the form:

(A-4) $\psi_n u = a_1^{(n)}\, \phi_1(u) + a_2^{(n)}\, \phi_2(u) + \cdots a_n^{(n)}\, \phi_n(u)$

Let us carry out this orthogonalization step by step. The first function $\psi_1(u)$ must have the form $c\,\phi_1(u)$.

Determining the constant c from the condition

$$\int_a^b \psi_1(u)\, \psi_1^*(u)\, du = 1$$

25

we find

$$(A-5) \quad \psi_1(u) = \frac{\phi_1(u)}{\left[\int_a^b \phi_1(u)\, \phi_1^*(u)\, du \right]^{\frac{1}{2}}}$$

Let us assume that the first n functions $\psi_1(u)$, $\psi_2(u)$, $\cdots$ $\psi_n(u)$ have been determined. The function $\psi_{n+1}(u)$ must be a linear combination of these and the function $\phi_{n+1}(u)$ in the form:

$$(A-6) \quad \psi_{n+1}(u) = c_1 \psi_1(u) + c_2 \psi_2(u) + \cdots + c_n \psi_n(u) + c\, \phi_{n+1}(u) .$$

We determine the constant c_i (i = 1, 2, $\cdots$ n) from the conditions of orthogonality of $\psi_{n+1}(u)$ to $\psi_1(u)$, $\psi_2(u)$ $\cdots$ $\psi_n(u)$; multiplying Eq. (A-6) by $\psi_1^*(u)$ and integrating we obtain, using the orthogonality and normality conditions:

$$(A-7a) \quad c_1 + c \int_a^b \phi_{n+1}(u)\, \psi_1^*(u)\, du = 0 .$$

Similarly, by multiplying by $\psi_2^*(u)$ and integrating:

$$(A-7b) \quad c_2 + c \int_a^b \phi_{n+1}(u)\, \psi_2^*(u)\, du = 0 ,$$

and so on. Finally we obtain

$$(A-7c) \quad c_n + c \int_a^b \phi_{n+1}(u)\, \psi_n^*(u)\, du = 0 .$$

Substituting c_1, c_2, $\cdots$ c_n from Eqs. (A-7) into Eq. (A-6) one obtains:

$$(A-8) \quad \psi_{n+1}(u) = c \left[\phi_{n+1}(u) - \sum_{j=1}^{n} \left(\int_a^b \phi_{n+1}(u)\, \psi_j^*(u)\, du \right) \psi_j(u) \right] .$$

26

The constant c can be determined from the normalization $\int \psi_{n+1} \psi_{n+1}^{*} \, du = 1$. Using this, the final result will be:

$$(A\text{-}9) \qquad \psi_{n+1}(u) = \frac{\phi_{n+1}(u) - \sum\limits_{j=1}^{n} \left(\int_{a}^{b} \phi_{n+1} \psi_{j}^{*} \, du \right) \psi_{j}(u)}{\left\{ \int_{a}^{b} \left| \phi_{n+1}(u) - \sum\limits_{j=1}^{n} \left(\int_{a}^{b} \phi_{n+1} \psi_{j}^{*} \, du \right) \psi_{j}(u) \right|^{2} \, du \right\}^{\frac{1}{2}}}$$

where $|A|^{2} = AA^{*}$.

By means of Eq. (A-9) the functions $\psi_{2}(u)$, $\psi_{3}(u) \cdots \psi_{n+1}(u)$ may be found in succession by cascade procedure.

After having constructed the orthonormal system, one can write, for any arbitrary function f(u), its series:

$$(A\text{-}10a) \qquad f(u) = \sum_{n=1}^{\infty} A_{n} \psi_{n}(u)$$

where the coefficients A_{n} may be found by using Eq. (A-3) to be

$$(A\text{-}10b) \qquad A_{n} = \int_{a}^{b} f(u) \psi_{n}^{*}(u) \, du$$

It is, of course, generally impossible to guarantee the convergence of the series in (A-10a). It would be, therefore, more correct to say that this series corresponds to the function f(u) rather than is equal to it.

Substituting Eq. (A-4) into Eq. (A-10a) one obtains:

$$(A\text{-}11) \qquad f(u) = \sum_{n=1}^{\infty} A_{n} \left[a_{1}^{(n)} \phi_{1}(u) + a_{2}^{(n)} \phi_{2}(u) + \cdots a_{n}^{(n)} \phi_{n}(u) \right]$$

f(u) is written in terms of combination of the original nonorthogonal set, $\phi_{n}(u)$.

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